University of California, Berkeley
Physics 110B, Spring 2004 (*Strovink*)

HINTS FOR ASSIGNED EXERCISES 42-59

42.
Working in the far zone $r' \ll \lambda \ll r$, consider azimuthally symmetric ($m = 0$) electric quadrupole ($E2_0$) radiation. At a particular angular frequency ω, work with the complex fields $\tilde{\vec{B}}(\vec{r})$ and $\tilde{\vec{E}}(\vec{r})$ defined by

$$\vec{B}(\vec{r},t) \equiv \mathrm{Re}\left(\tilde{\vec{B}}(\vec{r})e^{-i\omega t}\right)$$
$$\vec{E}(\vec{r},t) \equiv \mathrm{Re}\left(\tilde{\vec{E}}(\vec{r})e^{-i\omega t}\right) .$$

For E-type radiation, the magnetic field $\tilde{\vec{B}}$ ($\perp \hat{r}$) is proportional to the vector spherical harmonic $\vec{X}$:

$$\tilde{\vec{B}} \propto \vec{X}_{20}(\theta,\phi) \equiv \vec{L}\,Y_{20}(\theta,\phi) ,$$

with $i\vec{L} \equiv \vec{r}\times\nabla$. Use the fact that

$$\tilde{\vec{E}} \approx c\tilde{\vec{B}} \times \hat{r}$$

in the far zone. Obtain a function $f(\theta,\phi)$ such that the radiated power P in the far zone is proportional to it:

$$\frac{dP}{d\Omega} \propto f(\theta,\phi) .$$

Hint:
Since Y_{l0} is independent of ϕ, the only part of $\vec{r}\times\nabla$ that is relevant to this problem is $\hat{\phi}\frac{\partial}{\partial\theta}$ (see problem 40). This makes it easy to compute the angular dependence of $\tilde{\vec{B}}$. The time-averaged Poynting vector is

$$\langle\vec{S}\rangle = \frac{1}{2\mu_0}\mathrm{Re}(\tilde{\vec{E}} \times \tilde{\vec{B}}^*)$$

(see Griffiths Problem 11.15). Combine this with the equation

$$\tilde{\vec{E}} \approx c\tilde{\vec{B}} \times \hat{r}$$

and apply the $bac - cab$ rule. This should convince you that $P \propto |\tilde{\vec{B}}|^2$.

43.
At $t = 0$, charges $+e$ lie on the top right and bottom left corners of a square of side b in the xy plane that is centered at the origin; charges $-e$ lie on the top left and bottom right corners.
(a.)
Determine the lowest-l nonvanishing electrostatic multipole moment(s) of the charge distribution.
Hint:
Start with the definition

$$q_{lm} \equiv \int d\tau' \rho(\vec{r}\,')r'^l\,Y^*_{lm}(\theta',\phi')$$

of the multipole moments. For four point charges, the charge distribution is the sum of four 3-dimensional δ functions:

$$\frac{\rho(\vec{r}\,')}{e} = \sum_{n=0}^{3}(-1)^n\delta^3(\vec{r}\,' - \vec{a}_n) ,$$

where the four charges are located at spherical polar coordinates (r,θ,ϕ) of

$$\vec{a}_n = \left(\tfrac{b}{\sqrt{2}}, \tfrac{\pi}{2}, (2n+1)\tfrac{\pi}{4}\right) .$$

As usual, the integral over a δ function is equal to the value of the integrand where the δ function is infinite. Taking the dependence upon ϕ of Y^*_{lm} to be $\exp(-im\phi)$,

$$q_{lm} \propto \sum_{n=0}^{3}(-1)^n\exp\left(-im(2n+1)\tfrac{\pi}{4}\right) .$$

For what unique value of $|m|$ does the RHS sum to a nonzero result? Note that both signs of m contribute (with what relative weight?). What is the lowest-l spherical harmonic Y_{lm} for which this value of $|m|$ is possible? Is that particular

Y_{lm} nonzero at $\theta = \frac{\pi}{2}$ where the charges lie?
(b.)
The static charge distribution in (a.) now is set into oscillation: as time advances, the position vector of each charge is multiplied by the same factor $1 + \epsilon \cos \omega t$, where ω and $0 < \epsilon \ll 1$ are real constants. (Note that the charges do not move in a circle.) Using the fact that a static electric multipole corresponding to a given l and m, when caused to oscillate, yields E-type (TM) multipole radiation of the *same* l and m, what type(s) of radiation (*e.g.* $\mathrm{E1}_0$) is (are) emitted?
(c.)
Using the facts introduced in the previous problem, but generalizing them to the spherical harmonic(s) appropriate here, obtain a function $f(\theta, \phi)$ such that the radiated power P in the far zone $b \ll \frac{2\pi c}{\omega} \ll r$ is proportional to it:

$$\frac{dP}{d\Omega} \propto f(\theta, \phi) \ .$$

At how many points on the unit sphere (*e.g.* the north pole) does this radiation pattern vanish?
Hint:
In principle it would be acceptable to take advantage of the fact that Jackson (page 437) has tabulated the angular dependence of $|\vec{X}_{lm}|^2$ for $0 \le l \le 2$. [In the previous problem you already demonstrated your ability to compute an $\vec{X}_{lm}$ given a Y_{lm}.] But in this problem, $\vec{X}_{lm}$'s with *two* values of m contribute *coherently*; the $\vec{\tilde{B}}$'s corresponding to each $\vec{X}_{lm}$ must be added before the square modulus of $\vec{\tilde{B}}$ can be taken. This forces you to calculate the $\vec{X}_{lm}$'s that you need without taking advantage of Jackson's table. If you get the algebra right, you'll find that the radiation pattern takes a fairly simple form, but it vanishes at more than two points on the unit sphere.

44.
Griffiths Problem 11.15.
Hint:
The algebra for this problem can be simplified somewhat by defining $u \equiv \cos\theta$ and maximizing Griffiths' Eq. (11.74) with respect to u. Solve the resulting quadratic equation for u. In the ultrarelativistic limit it will also be convenient to define $\epsilon \equiv 1 - \beta$. In that limit $\epsilon \ll 1$ and, near where the maximum radiation is emitted, $\theta \ll 1$. To obtain Griffiths' approximate result you will need the Taylor expansions

$$\cos\theta \approx 1 - \tfrac{\theta^2}{2}$$
$$(1+\epsilon)^n \approx 1 + n\epsilon \ .$$

45.
Start from the expression derived in class for the energy radiated by an accelerating point charge per steradian per unit of *retarded* time t':

$$\frac{dW}{d\Omega\, dt'} = \left(\frac{q}{4\pi\epsilon_0}\right)^2 \frac{\epsilon_0}{c} \frac{\left|\hat{R} \times \left[(\hat{R} - \vec{\beta}) \times \dot{\vec{\beta}}\right]\right|^2}{(1 - \hat{R}\cdot\vec{\beta})^5} \ .$$

Consider *synchrotron radiation* by a particle of charge q moving in a circular orbit of radius b in a coordinate system where

$$\hat{\beta} = \hat{z}$$
$$\hat{\dot{\beta}} = \hat{x} \ ,$$

i.e. $\hat{x}$ points toward the center of the circle and $\hat{z}$ points along its circumference in the particle's direction of motion. Define

$$\hat{R} \equiv (n_x, n_y, n_z) \ ,$$

where $\hat{n}$ is a unit vector extending from the particle in an arbitrary direction towards which an element of radiation is emitted.
(a.)
Show that

$$\hat{R} \times \left[(\hat{R} - \vec{\beta}) \times \hat{\dot{\beta}}\right] = \hat{n} n_x - \hat{x} - \beta \hat{n} \times \hat{y} \ .$$

Hint:
Apply the $bac - cab$ rule.
(b.)
Using this result, show that

$$\left|\hat{R} \times \left[(\hat{R} - \vec{\beta}) \times \hat{\dot{\beta}}\right]\right|^2 = 1 - 2\beta n_z + \beta^2 n_z^2 - (1-\beta^2) n_x^2$$

(c.)
Consider a set of spherical polar coordinates centered at the particle (*not* at the center of the beam circle). Taking θ to be the polar angle of $\hat{n}$ relative to $\hat{z}$, and ϕ to be its azimuth about $\hat{z}$, express n_x and n_z in terms of θ and ϕ.
Hint:
Consult GIC #4.
(d.)
Using the results of (b.) and (c.), show that

$$\frac{dW}{d\Omega\, dt'} = \Big(\frac{q}{4\pi\epsilon_0}\Big)^2 \frac{\epsilon_0}{c} \times \\ \times \frac{\dot{\beta}^2}{(1-\beta\cos\theta)^3}\Big(1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\Big) .$$

46.
Consider the result of the previous problem in the relativistic limit $\gamma \gg 1$. In that limit, the nonnegligible part of the total radiation that is emitted occurs at polar angles θ such that $\gamma\theta$ is of order unity.
(a.)
Approximating $\cos\theta$ and $\sin\theta$ to lowest nonvanishing order in θ, perform the integration over $d\Omega = d(\cos\theta)\, d\phi$, integrating by parts where necessary, to show that

$$4\pi\epsilon_0 \frac{dW}{dt'} = \frac{2}{3c^3}\,(q\dot{\beta}c)^2\gamma^4 .$$

[Note that $(q\dot{\beta}c)^2$ is equivalent to $\ddot{p}^2$, where p is the electric dipole moment of the point charge relative to the origin. Therefore this result is the same as the (nonrelativistic) Larmor formula, except for the additional factor γ^4.]
Hint:
In the relativistic limit, the quantities $1/\gamma$ and θ (where most of the radiation is emitted) are small and of the same order. Using Taylor expansions, show that

$$1 - \beta\cos\theta \approx \frac{1+\gamma^2\theta^2}{2\gamma^2} .$$

Write

$$\oint d\Omega \approx \int_0^{2\pi} d\phi \int_0^b \theta\, d\theta ,$$

where $b \gg 1/\gamma$ includes the nonnegligible part of the radiation pattern (to lowest order your answer will not depend on its exact value). Integrating the first term of the integrand is trivial; the second term may be integrated by parts.

Alternatively, this problem may be done more easily *without* making any approximation! Define $u \equiv 1 - \beta\cos\theta$ and express the integrand as a function of u rather than θ. It becomes a polynomial in u that may trivially be integrated.
(b.)
In terms of the |momentum| P of the point charge and its rest mass m, show that

$$4\pi\epsilon_0 \frac{dW}{dt'} = \frac{2q^2}{3c^3}\,\frac{P^4}{m^4 b^2} ,$$

and thus that the power lost to synchrotron radiation depends on the fourth power of P, the inverse fourth power of m (making it usually negligible for all but electrons), and the inverse square of b.
Hint:
For motion around a circle of radius b, the centripetal acceleration $\dot{\beta}c$ is equal to $(\beta c)^2/b$. Use $P = \gamma\beta m c$.
(c.)
Suppose that you use an electron synchrotron that taxpayers can afford. It circulates highly relativistic electrons with $\beta \approx 1$. You want to build a new synchrotron with the same beam current, the same power lost to synchrotron radiation, but twice the beam momentum. Show that the radius b of the new synchrotron must increase by a factor of 16.
Hint:
Keep in mind that if the beam current stays the same while the synchrotron radius increases by a factor λ, the total number of radiating electrons increases by λ as well.

47.
A free-electron laser consists of a beam of electrons (with constant velocity βc) passing through a structure known as a *wiggler* or *undulator*. (These structures are used also in sections of a circular electron synchrotron such as the ALS.) Take the beam direction to be $\hat{z}$. Consider an alternating set of magnets (for compactness, these

are often permanent magnets, made of samarium cobalt as developed at LBL by the late Klaus Halbach). With a full period Δz, they produce a strong magnetic field that points alternately in the $+\hat{x}$ and $-\hat{x}$ directions.

(a.)

In the rest frame $\mathcal{S}'$ of the electron, with what fundamental angular frequency ω' does the magnetic field from the wiggler appear to oscillate?

Hint:

Apply a Lorentz contraction to get the apparent spacing of the magnets as seen by the electron. Divide it by the relative velocity (between the electron and the magnets) to obtain the time period of the magnetic field as seen by the electron.

(b.)

In $\mathcal{S}'$, the oscillating electron produces electromagnetic radiation with angular frequency ω'. Applying the relativistic Doppler shift to ("forward") radiation emitted along the beam direction, what angular frequency ω does that radiation have in the laboratory frame?

Hint:

As an alternative to the relativistic Doppler formula, you may simply perform an inverse Lorentz transformation on the 0^{th} component of the wave four-vector $(\omega/c, \vec{k})$. You are describing a massless photon, so $|\vec{k}| = \omega/c$ in either frame.

(c.)

Express λ, the wavelength of the forward radiation, as a multiple of Δz.

(d.)

At LBL's ALS, using an alternating set of magnets with $\Delta z = 10$ cm, an experimenter wishes to study the effect upon condensed-matter samples of a soft X-ray beam of wavelength 5 nm. Use this information to estimate the ALS beam energy (in GeV).

48.

Consider a medium with uniform fixed dielectric constant ϵ, permeability μ, and volume conductivity σ.

(a.)

Taking the curl of the two Maxwell equations which themselves involve the curl, and using Ohm's law ($\vec{J} = \sigma\vec{E}$) and the two other Maxwell equations where appropriate, derive the wave equations

$$\Big(\nabla^2 - \frac{\partial^2}{v^2\,\partial t^2}\Big)\vec{B} = \sigma\mu\frac{\partial\vec{B}}{\partial t}$$
$$\Big(\nabla^2 - \frac{\partial^2}{v^2\,\partial t^2}\Big)\vec{E} = \sigma\mu\frac{\partial\vec{E}}{\partial t} + \frac{1}{\epsilon}\nabla\rho_f\ ,$$

where the phase velocity^2 $v^2 \equiv \frac{1}{\epsilon\mu}$ and ρ_f is the volume free charge density.

Hint:

For example, to get the first equation (for $\vec{B}$), take the curl of Ampère's equation as modified by Maxwell:

$$\nabla\times\vec{H} = (\vec{J}_{(f)} = \sigma\vec{E}) + \frac{\partial\vec{D}}{\partial t}\ .$$

Consult Rule (11) on GIC #2 to transform the curl curl. Argue that $\nabla\cdot\vec{H} = 0$ because $\vec{B} = \mu\vec{H}$. Substitute $\vec{D} = \epsilon\vec{E}$ and use Faraday's law to re-express $\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$.

(b.)

In the wave equation for $\vec{B}$ derived in (a.), substitute

$$\vec{B}(\vec{r},t) = \mathrm{Re}\Big(\vec{\tilde{B}}\exp\big(i(\vec{\tilde{k}}\cdot\vec{r} - \omega t)\big)\Big)\ ,$$

where $\vec{\tilde{B}}$ and $\vec{\tilde{k}}$ are complex (vector) constants. Show that

$$\frac{\tilde{k}^2}{\mu\omega^2} = \epsilon(1 + i\beta)\ ,$$

where $\beta \equiv \frac{\sigma}{\epsilon\omega}$.

Hint:

After substituting for $\vec{B}$ in terms of $\vec{\tilde{B}}$, take advantage of the fact that the operators ∇ and $\frac{\partial}{\partial t}$ commute with the operator Re. Obtain a complex expression whose real (physical) part vanishes. Then define the imaginary (unphysical) part of that same expression also to vanish. Solve the resulting complex equation for $\tilde{k}^2$.

(c.)

Following Griffiths' notation, write $\tilde{k} \equiv k + i\kappa$, where k and κ are real. Show that

$$\frac{k}{\omega/v} = \sqrt{\frac{\sqrt{1+\beta^2}+1}{2}}$$
$$\frac{\kappa}{\omega/v} = \sqrt{\frac{\sqrt{1+\beta^2}-1}{2}}$$

are solutions to the equation that is the result of part (b.).

Hint:

Keep the algebra under control by defining $q \equiv \frac{\tilde{k}}{(\omega/v)}$ and solving the equation $q^2 = 1 + i\beta$. Write $q = r + is$, where r and s are real. Solve the imaginary part of the resulting equation for s in terms of β and r. Substituting for s, solve the resulting quadratic equation for r^2. Choose the sign for which r^2 is positive. Taking the square root, choose the sign for which $r = 1$ for $\beta = 0$ (a pure insulator). Returning to your result for s in terms of β and r, plug in your result for r and manipulate to get s.

(d.)

κ^{-1}, the inverse of the imaginary part of $\tilde{k}$, is called the *skin depth*. Show that the skin depth approaches

$$\sqrt{\frac{2}{\mu\sigma\omega}} \text{ when } \beta \gg 1 \text{ (good conductor)}$$
$$\frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}} \text{ when } \beta \ll 1 \text{ (poor conductor)} .$$

Hint:

For a poor conductor, use the Taylor expansion $(1 + \beta)^n \approx 1 + n\beta$.

49.

Please refer to the notation and results of the previous problem.

(a.)

At normal incidence at the interface between two dissimilar materials 1 and 2, the (complex) electric field amplitude reflected back into material 1 is expressed as a (complex) ratio $\tilde{\mathcal{R}}$ to the (complex) incident amplitude. By matching boundary conditions for the electric and magnetic fields, $\tilde{\mathcal{R}}$ is routinely found to be given by the standard result

$$\tilde{\mathcal{R}} = \frac{\tilde{Z}_1^{-1} - \tilde{Z}_2^{-1}}{\tilde{Z}_1^{-1} + \tilde{Z}_2^{-1}} ,$$

where

$$\tilde{Z}^{-1} \equiv \frac{\tilde{k}}{\mu\omega} ,$$

the ratio of $\tilde{H}$ to $\tilde{E}$, is the medium's (complex) *admittance*. Consider the case in which material 1 is an insulator and material 2 is a conductor. If material 2 is an *excellent* conductor ($\beta \gg 1$), show that $\tilde{\mathcal{R}} \to -1$ regardless of the (finite) values taken by $\epsilon_{1,2}$ and $\mu_{1,2}$. Therefore *metals are shiny*.

(b.)

Suppose instead that material 2 is a *poor* conductor ($\beta \ll 1$) (otherwise all the conditions of part (a.) apply). Suppose further that, if both materials had zero conductivity, they would have equal admittance ($\sqrt{\epsilon_1/\mu_1} = \sqrt{\epsilon_2/\mu_2}$). Show that $\tilde{\mathcal{R}} \to -i\beta/4$.

(c.)

In a relatively more microscopic and detailed treatment, one assumes that N valence electrons per m^3 having charge $-e$ and mass m move in a potential well with effective spring constant $m\omega_0^2$ and damping coefficient γm. One defines the *complex dielectric constant* $\tilde{\epsilon}$ via

$$\frac{\tilde{\epsilon}}{\epsilon_0} - 1 \equiv \frac{\tilde{P}}{\epsilon_0 \tilde{E}} ,$$

where $\tilde{P}$ is the complex polarization, defined analogously to the complex electric and magnetic fields in the previous problem. For not-too-dense media in which the electric field felt by the electron is approximately the same as the average field, it is straightforward to solve the force equation for these oscillating electrons and determine the complex polarization $\tilde{P}$ they create. One obtains

$$\frac{\tilde{\epsilon}}{\epsilon_0} - 1 = \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega} ,$$

where the *plasma frequency*^{2} is

$$\omega_p^2 \equiv \frac{Ne^2}{m\epsilon_0} .$$

The complex dielectric constant includes the effects of all electrons (free and bound). Considering the result of part (b.) of the previous problem, the complex dielectric constant is related to the ordinary dielectric constant ϵ (which includes the effects only of bound electrons) by

$$\tilde{\epsilon} = \epsilon(1 + i\beta) = \frac{\tilde{k}^2}{\mu\omega^2} .$$

Represent a *good conductor* by $\omega_0 = 0$ (unbound) and $\gamma \gg \omega$ (overdamped). Using these results, show that the conductivity σ is approximately

$$\sigma \approx \frac{\epsilon_0 \omega_p^2}{\gamma} ,$$

i.e. measuring the low-frequency conductivity is a simple way to determine the damping coefficient.

Hint:

Consider the imaginary part of $\tilde{\epsilon}/\epsilon_0$.

(d.)

Represent the *ionosphere* by $\omega_0 = 0$ (unbound), and $\gamma \ll \omega$ (underdamped). Specialize to AM radio waves, for which $\omega < \omega_p$. Show that $|\tilde{\mathcal{R}}| \approx 1$, *i.e.* that AM radio waves are nearly fully reflected by the ionosphere. (At dusk, the ionosphere drops to sufficiently low altitude that reflection off it enables AM stations hundreds of miles away to be received.)

Hint:

Using the information given in this and the previous problem, first show that

$$\tilde{k}^2 = \mu\omega^2\epsilon_0\big(1 + \frac{\omega_w^2}{\omega_0^2 - \omega^2 - i\gamma\omega}\big) .$$

Then, for the underdamped case and for $\omega < \omega_p$, argue that $\tilde{k}$ is almost pure imaginary, causing the numerator of $\tilde{\mathcal{R}}$ to be the complex conjugate of the denominator.

50.

Griffiths Problem 9.11.

Hint:

Take the time average of $\frac{1}{2}\,\mathrm{Re}(\tilde{f}\tilde{g}^*)$ at the same point $\vec{r}$ over one period $T = 2\pi/\omega$.

51. Jones vectors.

For a plane transverse wave propagating in the $\hat{z}$ direction through a (not necessarily insulating) material with constant ϵ and μ, a (co)sinusoidal solution is represented by

$$\vec{E}(\vec{r},t) = \mathrm{Re}\Big(\vec{E}_0(x,y)\, e^{i(\tilde{k}z - \omega t)}\Big)$$
$$\vec{H}(\vec{r},t) = \mathrm{Re}\Big(\vec{H}_0(x,y)\, e^{i(\tilde{k}z - \omega t)}\Big) ,$$

where $\tilde{k}$ is the (not necessarily real) "wave vector" – here a scalar because we know it is directed along $\hat{z}$. Faraday's law causes $\vec{H}_0$ to be completely determined by $\vec{E}_0$:

$$\begin{aligned}\vec{H}_0 &\equiv \tilde{Z}^{-1}\hat{z}\times\vec{E}_0 \\ &= \frac{\tilde{k}}{\mu\omega}\hat{z}\times\vec{E}_0 ,\end{aligned}$$

so we focus on $\vec{E}_0$ as the sole independent variable. For a transverse wave $\vec{E}_0$ has no z component. Here we assume that the phase relationship between E_{0x} and E_{0y} is *fixed* – the wave is *fully polarized*. Then $\vec{E}_0$ is a complex transverse vector, completely specified by four components. In the Jones convention, all information carried by $\vec{E}_0$ except for its magnitude is written as a 2×1 column vector with the x component on top:

$$\begin{aligned}\vec{E}_0 &= \begin{pmatrix} E_{0x} \\ E_{0y} \end{pmatrix} \\ &\equiv \frac{1}{\sqrt{|\alpha^2| + |\beta^2|}} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} |\vec{E}_0| \\ &\equiv \vec{J}\,|\vec{E}_0| ,\end{aligned}$$

where $\vec{J}$ is the *Jones vector*. Jones vectors are defined only within an overall phase (because the absolute phase of an optical-frequency EM wave can't conveniently be measured); therefore one has the freedom to set α equal to unity (unless it vanishes, in which case β is set to unity). The above form involving the complex constants α and β is a general Jones vector, corresponding to elliptical polarization. More common Jones vectors are

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \quad ,$$

corresponding, respectively, to linear x, linear y, RH circular, and LH circular polarization.

(a.)
At $z = 0$, show (counterintuitively!) that the electric field vector for RH polarized light precesses *clockwise* around $\hat{z}$, *i.e.* it precesses according to the LH rule.
Hint:
Evaluate $\mathrm{Re}\big(\vec{J}e^{-i\omega t}\big)$ and examine the time evolution of the resulting x and y components.
(b.)
Suppose that a particular state of elliptical polarization has nonvanishing x and y electric field components. Then, within an arbitrary overall phase, it may be represented by the Jones vector

$$\vec{J}_1 = \frac{1}{\sqrt{1+|\beta|^2}}\begin{pmatrix}1\\ \beta\end{pmatrix} ,$$

where β is a complex constant. You wish to characterize this state of polarization as "RH elliptical" or "LH elliptical", depending on whether (at $z = 0$) the electric field vector precesses clockwise or counterclockwise around $\hat{z}$. What property of β would you use to decide whether this state is RH or LH elliptical?
Hint:
Again evaluate $\mathrm{Re}\big(\vec{J}_1 e^{-i\omega t}\big)$ and examine the time evolution of the resulting x and y components. If the y component becomes negative as t increases from 0, the elliptical polarization is right-handed; if it becomes positive, the polarization is left-handed. What property of β controls this behavior?
(c.)
For the conditions of part (b.), decompose $\vec{J}_1$ into a linear sum (with real coefficients) of a wave with linear polarization plus a wave with RH circular polarization. Perform this same task with "RH" replaced by "LH". If you are successful in both tasks, you might wonder whether there really exists a unique association of RH or LH behavior with $\vec{J}_1$. Would this concern invalidate your answer to (b.)?
Hint:
For example, to decompose $\vec{J}_1$ into a linearly polarized wave plus a RHCP wave, take the difference

$$\begin{pmatrix}1\\ \beta\end{pmatrix} - C\begin{pmatrix}1\\ -i\end{pmatrix} .$$

Determine C such that this difference (the linearly polarized part) has elements that both have the same phase.

52. Irradiance and Jones vectors.
Consider two transverse plane waves A and B which move in vacuum and are combined together (*i.e.* by a Michelson interferometer). The beams have complex electric fields

$$\begin{pmatrix}E^A_{0x}\\ E^A_{0y}\end{pmatrix} = \frac{|\vec{E}^A_0|}{\sqrt{|\alpha|^2+|\beta|^2}}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$
$$\begin{pmatrix}E^B_{0x}\\ E^B_{0y}\end{pmatrix} = \frac{|\vec{E}^B_0|}{\sqrt{|\gamma|^2+|\delta|^2}}\begin{pmatrix}\gamma\\ \delta\end{pmatrix} .$$

Express the combined irradiance

$$I_{A+B} \equiv \langle \vec{S}_{A+B}\cdot\hat{z}\rangle ,$$

where $\vec{S}$ is the Poynting vector and $\langle\ \rangle$ is a time average, as a function of the complex constants α, β, γ, δ, and the uncombined irradiances I_A and I_B of the individual beams.
Hint:
Start from the result of Griffiths' Problem 9.11, modified so that it is true in material as well as vacuum:

$$\langle\vec{S}\rangle = \frac{1}{2\mu}\mathrm{Re}(\tilde{\vec{E}}\times\tilde{\vec{B}}^*) .$$

Express Faraday's law in terms of the complex fields so that you can write $\tilde{\vec{B}}^*$ in terms of $\tilde{\vec{E}}^*$. You should obtain

$$\tilde{\vec{B}}^* = \frac{\tilde{k}^*}{\omega}\hat{z}\times\tilde{\vec{E}}^* .$$

By definition, the irradiance $I \equiv \langle\vec{S}\cdot\hat{z}\rangle$. For the individual beams, you should obtain

$$\begin{aligned} I_{A,B} &= \frac{1}{2\mu\omega}\mathrm{Re}\big(\tilde{k}^*\,\tilde{\vec{E}}_{A,B}\cdot\tilde{\vec{E}}^*_{A,B}\big)\\ &= \frac{\mathrm{Re}\,\tilde{k}}{2\mu\omega}|\tilde{\vec{E}}_{A,B}|^2 . \end{aligned}$$

For the combined beam (C), write $\tilde{\vec{E}}_C = \tilde{\vec{E}}_A + \tilde{\vec{E}}_A$. You should obtain

$$\begin{aligned} I_C &= I_A + I_B + \frac{1}{2\mu\omega}\mathrm{Re}\big(\tilde{k}^*(\tilde{\vec{E}}_A\cdot\tilde{\vec{E}}^*_B+\tilde{\vec{E}}_B\cdot\tilde{\vec{E}}^*_A)\big)\\ &= I_A + I_B + \frac{\mathrm{Re}\,\tilde{k}}{\mu\omega}\mathrm{Re}\big(\tilde{\vec{E}}_A\cdot\tilde{\vec{E}}^*_B\big) . \end{aligned}$$

Finally, substitute the Jones vectors supplied for $\vec{\tilde{E}}_A$ and $\vec{\tilde{E}}_B$:

$$\vec{\tilde{E}}_{A,B} = |\vec{\tilde{E}}_{A,B}|\, \vec{J}_{A,B}\ .$$

Solve for I_C as a function of the variables required by the problem.

53.
(a.)
A set of N ideal linear polarizers $\mathrm{L}_1 \ldots \mathrm{L}_N$ is arranged so that $\hat{x}$ polarized light passes through them in ascending order. The transmission axis of polarizer n is oriented along $(\hat{x}\cos\phi_n + \hat{y}\sin\phi_n)$, where $\phi_n = \frac{\pi n}{2N}$. In the limit $N \to \infty$, deduce the Jones matrix for this set.
Hint:
Consider the effect on $\hat{x}$ polarized light of the first polarizer only. Use Pedrotti×2 ($\equiv$ P×2) Eq. (14-15) with $\theta = \pi/2N$. Taking $N \gg 1$, show that the fractional reduction in electric field amplitude is only of order N^{-2}. In the limit $N \to \infty$, argue that the total reduction in electric field amplitude after the infinite set of ideal polarizers is negligible. Considering the effect of this set of polarizers upon $\hat{x}$ polarized light, what is the left-hand column of the corresponding Jones matrix? What about the right-hand column?
(b.)
Consider a *twisted nematic cell*, as found in an LCD display. It functions as a *rotator* (P×2 Eq. (14-21)). Show that if the rotator parameter $\beta = \frac{\pi}{2}$, the twisted cell will have the same effect on $\hat{x}$ polarized light as does the set of polarizers described in (a.).
(c.)
Do the devices in (a.) and (b.) also have equivalent effect on $\hat{y}$ polarized light? Explain.
Hint:
Their effect is equivalent iff their Jones matrices are the same.

54.
Apart from an experimentally irrelevant overall phase, an ideal wave plate of thickness D with phase retardation difference

$$\delta \equiv (n_x - n_y)\frac{\omega D}{c}\ ,$$

having its slow axis along $\hat{x}$, is represented by the Jones matrix

$$M_{\mathrm{W}}(\phi = 0) = \begin{pmatrix} e^{i\delta/2} & 0 \\ 0 & e^{-i\delta/2} \end{pmatrix}\ .$$

If instead the wave plate has its slow axis along $(\hat{x}\cos\phi + \hat{y}\sin\phi)$, show that it is represented by the general Jones matrix

$$M_{\mathrm{W}}(\phi) = \begin{pmatrix} \cos\frac{\delta}{2} + i\sin\frac{\delta}{2}\cos 2\phi & i\sin\frac{\delta}{2}\sin 2\phi \\ i\sin\frac{\delta}{2}\sin 2\phi & \cos\frac{\delta}{2} - i\sin\frac{\delta}{2}\cos 2\phi \end{pmatrix}$$

Note that $\delta = \frac{\pi}{2}$ for a quarter-wave plate (QWP) and $\delta = \pi$ for a half-wave plate (HWP), which is equivalent to two QWPs. Note also that, like the general Jones matrix $M_{\mathrm{L}}(\phi)$ for the ideal linear polarizer (P×2 Eq. (14-15)), $M_{\mathrm{W}}(\phi)$ is *symmetric* and invariant to the transformation $\phi \to \phi + \pi$. However, unlike $M_{\mathrm{L}}(\phi)$, $M_{\mathrm{W}}(\phi)$ is also *unitary* ($M^{-1} = M^\dagger$) with unit determinant, preserving the irradiance.
Hint:
First make a 2D coordinate (passive) rotation $(x, y) \to (x', y')$ so that the wave plate's slow axis is along $\hat{x}'$. Then apply the Jones matrix $M_{\mathrm{W}}(\phi = 0)$. Finally, rotate back to the (x, y) frame.

55.
Use the result of the previous problem to do Pedrotti×2 Problem 14-11. To get their result you must assume, as they do [Eqs. (14-17)-(14-20)], that the wave plate's slow axis lies along either the x or y axis.
Hint:
The initial state is described by P×2 Eq. (14-6). After the final state is put in the same form (with α replaced by α'), the final angle of inclination is simply α'. Alternatively, P×2 Eq. (14-10) supplies a general result for the final angle of inclination, useful even for elliptical polarization; this problem involves only the linearly polarized case for which $\epsilon = 0$ in that equation.

56.
(a.)
Do Pedrotti×2 Problem 14-17. Does their Jones matrix really convert *any* state of incident polarization to a *finite* irradiance of RH polarized light? Explain.
(b.)
Devise a combination of ideal wave plate(s) and polarizer(s) that, within a multiplicative constant, yields the Jones matrix of part (a.). Supply the absolute magnitude of this constant. Congratulations! You have invented an ideal *homogeneous right-hand circular polarizer*.
Hint:
As an alternative to multiplying random Jones matrices until you find a combination that works, you could reason physically. You want a device that emits only RH circularly polarized (RHCP) light. Recall the "points-of-the-compass" diagram drawn in class: applying a QWP with slow axis at $+45°$ produces the transitions $\hat{x}$ LP $\rightarrow$ LHCP $\rightarrow$ $\hat{y}$ LP $\rightarrow$ RHCP $\rightarrow$ $\hat{x}$ LP (replace $\rightarrow$ by $\leftarrow$ for a QWP with slow axis at $-45°$). Therefore, for example, an $\hat{x}$ linear polarizer upstream of a QWP with slow axis at $-45°$ emits only RHCP light. However, this isn't the complete answer, because you want a device that emits *some* RHCP light unless the incident beam is LHCP (see part (a.)). What can you put upstream of the $\hat{x}$ linear polarizer that emits *some* $\hat{x}$ LP light unless the beam incident upon it is LHCP?
(c.)
Show that the result of part (b.) functions also as a *right-hand circular analyzer*, *i.e.* it fully transmits RH circularly polarized light and fully absorbs LH circularly polarized light.

57. Stokes vectors #1.
Using the standard definition of the complex electric field $\vec{E}_0$,

$$\vec{E}(z,t) = \mathrm{Re}\big(\vec{E}_0 \exp\left(i(\tilde{k}z - \omega t)\right)\big) ,$$

consider the case in which the phase difference between its x and y components

$$\epsilon(t) = \arg E_{0x} - \arg E_{0y}$$

is not necessarily fixed, as would be the case for fully polarized light, but rather is allowed to vary with time – slowly with respect to ω^{-1}, but rapidly with respect to experimenters' ability to measure it. The *Stokes vector* $\mathcal{S}$ is defined by the real elements

$$\mathcal{S} \equiv \begin{pmatrix} \mathcal{S}_0 \\ \mathcal{S}_1 \\ \mathcal{S}_2 \\ \mathcal{S}_3 \end{pmatrix} \equiv \frac{\mathrm{Re}\,\tilde{k}}{\mu\omega} \begin{pmatrix} |E_{0x}|^2 + |E_{0y}|^2 \\ |E_{0x}|^2 - |E_{0y}|^2 \\ \langle 2\,\mathrm{Re}(E_{0x}E_{0y}^*)\rangle \\ \langle 2\,\mathrm{Im}(E_{0x}E_{0y}^*)\rangle \end{pmatrix} ,$$

where $\langle\rangle$ denotes the time average.
(a.)
Show that

$$\mathcal{S} = \frac{\mathrm{Re}\,\tilde{k}}{\mu\omega} \begin{pmatrix} |E_{0x}|^2 + |E_{0y}|^2 \\ |E_{0x}|^2 - |E_{0y}|^2 \\ \langle 2|E_{0x}||E_{0y}|\cos\epsilon\rangle \\ \langle 2|E_{0x}||E_{0y}|\sin\epsilon\rangle \end{pmatrix} .$$

Hint:
Without loss of generality, substitute

$$E_{0x} = |E_{0x}|\, e^{i(\eta(t) - \epsilon(t)/2)}$$
$$E_{0y} = |E_{0y}|\, e^{i(\eta(t) + \epsilon(t)/2)}$$

in the definition of $\mathcal{S}$.
(b.)
The *normalized Stokes vector* $\bar{\mathcal{S}}$ is defined to be the usual Stokes vector divided by $\mathcal{S}_0$, so that its topmost element is unity. Consider a fully polarized beam in an arbitrary state of polarization described by the general Jones vector

$$J = \frac{1}{\sqrt{|\alpha|^2 + |\beta|^2}} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} .$$

Show that the normalized Stokes vector for this beam is

$$\bar{\mathcal{S}} = \frac{1}{|\alpha|^2 + |\beta|^2} \begin{pmatrix} |\alpha|^2 + |\beta|^2 \\ |\alpha|^2 - |\beta|^2 \\ 2\,\mathrm{Re}(\alpha\beta^*) \\ 2\,\mathrm{Im}(\alpha\beta^*) \end{pmatrix} .$$

Hint:
For a fully polarized beam, $\epsilon(t)$ in the result of part (a.) becomes a constant equal to $\arg(\beta) - \arg(\alpha)$.

(c.)
Using the result of (b.) and your knowledge of Jones vectors, show that fully linearly polarized beams in the $\hat{x}$, $\hat{y}$, $\frac{1}{\sqrt{2}}(\hat{x}+\hat{y})$, and $\frac{1}{\sqrt{2}}(\hat{x}-\hat{y})$ directions are described, respectively, by the normalized Stokes vectors

$$\begin{pmatrix}1\\1\\0\\0\end{pmatrix}\quad\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}\quad\begin{pmatrix}1\\0\\1\\0\end{pmatrix}\quad\begin{pmatrix}1\\0\\-1\\0\end{pmatrix},$$

and that fully circularly RH and LH polarized beams are described, respectively, by the normalized Stokes vectors

$$\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\quad\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}.$$

58. Stokes vectors #2.
Please refer to the notation and results of the previous problem.
(a.)
For fully polarized ("p") light (ϵ fixed), show that

$$\mathcal{S}_1^2+\mathcal{S}_2^2+\mathcal{S}_3^2=\mathcal{S}_0^2\ .$$

Hint:
Use the result of part (a.) of the previous problem; with ϵ fixed, it is unnecessary to average over time.
(b.)
Natural ("n") light is completely unpolarized. It has $|E_{0x}|=|E_{0y}|$, but the phases of both E_{0x} and E_{0y} vary randomly with time so that $\langle\cos\epsilon\rangle=\langle\sin\epsilon\rangle=0$. For natural light, show (conversely to (a.)) that

$$\mathcal{S}_1=\mathcal{S}_2=\mathcal{S}_3=0\ .$$

59. Stokes vectors #3.
Please refer to the notation and results of the previous two problems. Consider four devices: (A) a grey filter passing half the incident irradiance; (B) an $\hat{x}$ polarizer; (C) an $\frac{1}{\sqrt{2}}(\hat{x}+\hat{y})$ polarizer; (D) a RH circular analyzer. After passing through (only) device X, the beam has irradiance I_{X}. It can be shown that

$$\mathcal{S}=2\begin{pmatrix}I_{\mathrm{A}}\\I_{\mathrm{B}}-I_{\mathrm{A}}\\I_{\mathrm{C}}-I_{\mathrm{A}}\\I_{\mathrm{D}}-I_{\mathrm{A}}\end{pmatrix}.$$

Therefore, a Stokes vector can be completely determined by *measuring only irradiances*. This reveals one extent to which Stokes "vectors" satisfy vector properties. The additive property normally associated with a vector, $\mathcal{S}_{\mathrm{tot}}=\mathcal{S}_A+\mathcal{S}_B$ for two beams A and B, holds only if their *irradiances* rather than their amplitudes add, *i.e.* only if the two beams are *completely mutually incoherent*. This is a total contrast to Jones vectors, which can be defined only for fully polarized beams and can be added only if the two beams are completely mutually coherent.
(a.)
Using the additive property for Stokes vectors in mutually incoherent beams, show that an arbitrary beam

$$\mathcal{S}=\begin{pmatrix}\mathcal{S}_0\\\mathcal{S}_1\\\mathcal{S}_2\\\mathcal{S}_3\end{pmatrix}$$

is the (necessarily incoherent) superposition of a fully polarized beam p and a natural-light beam n. Show this by specifying the elements of the constituent Stokes vectors $\mathcal{S}_p$ and $\mathcal{S}_n$ in terms of the elements of the overall Stokes vector $\mathcal{S}$.
Hint:
The bottom three elements of $\mathcal{S}_n$ vanish, so the bottom three elements of $\mathcal{S}_p$ are the same as those of $\mathcal{S}$. Using the result of part (a.) of the previous problem, deduce the top element of $\mathcal{S}_p$.

(b.)
Define the *degree of polarization* V by

$$V \equiv \frac{I_p}{I_p + I_n} \ .$$

For the above arbitrary beam, show that

$$V = \frac{\sqrt{\mathcal{S}_1^2 + \mathcal{S}_2^2 + \mathcal{S}_3^2}}{\mathcal{S}_0} \ .$$

Hint:
From the original definition of $\mathcal{S}$, one sees that its top element is simply the irradiance of the beam.

Appendix: Mueller matrices
The *Mueller matrices* manipulate Stokes vectors in the same way that Jones matrices manipulate Jones vectors. For an $\hat{x}$ polarizer and for a $\frac{1}{\sqrt{2}}(\hat{x} + \hat{y})$ polarizer, the Mueller matrices are, respectively,

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ .$$

The Mueller matrices for a $\hat{y}$ polarizer and for a $\frac{1}{\sqrt{2}}(\hat{x} - \hat{y})$ polarizer are, respectively,

$$\frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ .$$

For a QWP with slow axis along x and for a homogeneous right-hand circular polarizer, the Mueller matrices are, respectively,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \ .$$

The Mueller matrices for a QWP with slow axis along y and for a homogeneous left-hand circular polarizer are, respectively,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \ .$$